Symplectic Dirac Operators for LA and GAHA
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§0 Introduction.

- Let F field, $\mathrm{char}(F) = 0$, $\epsilon \in \{\pm 1\}$ and $(\mathcal{V}, \beta_\epsilon)$ a non-deg. F-v. space.
- Consider $\mathcal{C}\ell_\epsilon := \mathcal{C}\ell(\mathcal{V}, \beta_\epsilon) = T(\mathcal{V})/I(\beta_\epsilon)$ with
$$I_\epsilon = \langle \{u \otimes v + \epsilon v \otimes u - \beta_\epsilon(u, v)1 \; ; \; u, v \in \mathcal{V}\}\rangle.$$
$\epsilon = +$: orthogonal Clifford, $\epsilon = -$: symplectic Clifford (Weyl algebra).
- $\mathcal{C}\ell_\epsilon$ not graded but there are linear isomorphisms
$$Q_\epsilon : \begin{cases} E^\bullet(\mathcal{V}) & \to \quad \mathcal{C}\ell_+ \\ S^\bullet(\mathcal{V}) & \to \quad \mathcal{C}\ell_- \end{cases}, \qquad Q_\epsilon(v_1 \cdots v_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \epsilon^{|\sigma|} v_{\sigma(1)} \cdots v_{\sigma(p)}.$$
Have $Q_\epsilon(\deg 2) = \mathfrak{so}(\mathcal{V}, \beta_+)$, if $\epsilon = +$ or $\mathfrak{sp}(\mathcal{V}, \beta_-)$, if $\epsilon = -$.
- Suppose A is a k-algebra (unital, assoc.) with $\mathcal{V} \subseteq A$.
Slogan: Rep. Theory of $A \otimes \mathcal{C}\ell_\epsilon \rightsquigarrow$ info on Rep. Theory of A.
- $\epsilon = +$, orthogonal case:
  - realization of DS, Vogan conjecture, Howe dualities, etc.
  - $D \in A \otimes \mathcal{C}\ell_+$ as the 'identity'.
  - the square $D^2 = \frac{1}{2}[D, D]_+$ is 'central' to the applications.
- What happens fo $\epsilon = -$?
  - a pair (D^+, D^-) of Dirac's and $[D^+, D^-]_-$ takes the place of D^2.

§1 Preliminaries on symplectic spinors.

- From now on $\beta_- = \omega$ and $\mathcal{C}\ell_- = \mathcal{W}$.
- Define $\pi, \iota : \mathcal{V} \to \mathrm{End}(S^\bullet(\mathcal{V}))$ by extending the action on $w \in \mathcal{V}$
$$\pi(v)(w) = vw, \qquad \iota(v)(w) = \omega(v, w) \text{ (as a derivation).}$$
Extend to $\mathcal{W}$ the map $\gamma(v) = \pi(v) + \frac{1}{2}\iota(v)$, yielding an inverse to Q:
$$\eta : \mathcal{W} \to S^\bullet(\mathcal{V}), \qquad \eta(x) = \gamma(x)(1).$$

- For $x \in \mathcal{W}$, write $x = \sum_d (x)_d$ for the decomp of $\eta(x) \in \oplus_d S^\bullet(\mathcal{V})$.
- Write $\mathcal{V} = V \oplus V'$ for a pair of max'l ω-isotropic subspaces.
Can assume (V, B) quadratic and identify $V' = V^*$ via B.
- The symplectic spinor module is $\mathcal{S} = S^\bullet(V)$ with $m : \mathcal{W} \to \mathrm{End}(\mathcal{S})$,
$$m(v)p = vp, \qquad m(\lambda)p = \partial_\lambda(p), \qquad \forall v \in V, \lambda \in V^*.$$
- Fix $\{v_j\} \subset V$ basis $\rightsquigarrow$ bases $\{v_i^*\}, \{v^i\}$ s.t. $B(v^i, v_j) = \delta_{ij} = v_i^*(v_j)$.
- Have $\mathfrak{sl}(2, F) = \langle H, R, L\rangle$, with $L = \frac{1}{2}\Delta, R = -\frac{1}{2}r^2$ and dual pair $(\mathfrak{sl}(2, F), \mathfrak{so}(V, B))$ inside $Q(S^2(\mathcal{V})) = \mathfrak{sp}(\mathcal{V}, \omega)$. Let $\nu : \mathfrak{so}(V, B) \hookrightarrow \mathcal{W}$.

§2 Lie algebras.

- Let $(\mathfrak{g}, B)$ be a quadratic LA with $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.
Example: $\mathfrak{g} = \mathfrak{sl}(2, F)$, $B(X, Y) = \frac{1}{2}\mathrm{Tr}(XY), \mathfrak{k} = \langle H\rangle, \mathfrak{p} = \langle R, L\rangle$.
- Let $A = \mathcal{U}(\mathfrak{g})$, $\mathcal{W} = \mathcal{W}(\mathfrak{p} \oplus \mathfrak{p}^*, \omega)$, with $\omega(\mathfrak{p}, \mathfrak{p}) = \{0\} = \omega(\mathfrak{p}^*, \mathfrak{p}^*)$ and
$$\omega(\lambda, v) = \lambda(v), \qquad \forall \lambda \in \mathfrak{p}^*, v \in \mathfrak{p}.$$
- Fix bases $\{z_i\} \subset \mathfrak{p}$ and $\{w_k\} \subset \mathfrak{k}$, then
$$\mathrm{End}(\mathfrak{p}) \cong \begin{cases} \mathfrak{p} \otimes \mathfrak{p} & \rightsquigarrow \quad D^+ = \sum_i z_i \otimes z^i, \\ \mathfrak{p} \otimes \mathfrak{p}^* & \rightsquigarrow \quad D^- = \sum_i z_i \otimes z_i^*. \end{cases}$$
- Consider $\nu_a : (\mathfrak{k} \xrightarrow{\mathrm{ad}} \mathfrak{so}(\mathfrak{p}, B) \hookrightarrow \mathcal{W})$ and the diagonal embedding
$$\Delta_\nu : \mathcal{U}(\mathfrak{k}) \to \mathcal{U}(\mathfrak{k}) \otimes \mathcal{W}, \qquad \Delta_\nu(x) = x \otimes 1 + 1 \otimes \nu_a(x).$$
- **Thm**(C-DM-M): In $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{W}$ we have
$$[D^+, D^-] = -(\Omega(\mathfrak{g}) - \tfrac{3}{2}\Omega(\mathfrak{k})) \otimes 1 + \tfrac{1}{2} \otimes \nu_a(\Omega(\mathfrak{k})) - \tfrac{1}{2}\Delta_\nu(\Omega(\mathfrak{k})).$$
- **Thm**(C-DM-M): Have $\eta(\nu_a(\Omega(\mathfrak{k})) \in S^4 \oplus S^0$ and when $F = \bar{F}$
$$(\nu_a(\Omega(\mathfrak{k}))_0 = 2(B(\rho_\mathfrak{k}, \rho_\mathfrak{k}) - B(\rho_\mathfrak{g}, \rho_\mathfrak{g})).$$
Further, $(\nu_a(\Omega(\mathfrak{k}))_4 \neq 0$, unless $B(\mathfrak{p}^{(1)}, \mathfrak{p}^{(1)}) = 0$.

- **Rem** $(\nu_a(\Omega(\mathfrak{k}))_4$ is the obstruction to LSA structure on $\mathfrak{k} \oplus (\mathfrak{p} \oplus \mathfrak{p}^*)$.

§3 Graded Affine Hecke algebras.

- Let $(V_0^*, \Phi, V_0, \Phi^\vee)$ be an irred. root system, $W = W(\Phi) \subset O(V_0, B_0)$ and $V = V_0 \otimes \mathbb{C}$, B bilinear extension of B_0. Fix $k : \Phi/W \to \mathbb{C}$.
- Recall $\mathbb{H} = \mathbb{H}(\Phi, k)$ is the quotient of $T(V) \rtimes \mathbb{C}[W]$ by

$$[u, v] = 0, \quad vs_\alpha - s_\alpha v = k(\alpha)\alpha(v), \quad \forall u, v \in V, \alpha \in \Phi.$$

- Let $A = \mathbb{H}$ and $\mathcal{W} = \mathcal{W}(V + V^*, \omega)$, $\omega(\lambda, v)$ with $\lambda \in V^*, v \in V$.
- Fix a basis $\{v_i\} \subset V_0$ and for $v \in V$ let

$$\widetilde{v} = v - T_v, \qquad T_v = \tfrac{1}{2} \sum_{\alpha > 0} k(\alpha)\alpha(v) s_\alpha$$

  Then, $\mathbb{H}$ is generated by $\{\widetilde{v_i}, s_\alpha\}$ with $w\widetilde{v_i} = \widetilde{w(v_i)}w$, $\forall w \in W$ and

$$[\widetilde{v_i}, \widetilde{v_j}] = \tfrac{1}{4} \sum_{(\alpha \prec \beta)} k(\alpha)k(\beta)A_{ij}(\alpha, \beta)[s_\alpha, s_\beta],$$

  with $A_{ij}(\alpha, \beta) = \alpha(v_i)\beta(v_j) - \alpha(v_j)\beta(v_i)$.

- **Lem**(C-DM-P): $[s_\alpha, s_\beta] \in \mathfrak{so}(V, B)$ and $\{[s_\alpha, s_\beta]\}_{\alpha,\beta}$ span $\mathfrak{so}(V, B)$.
  Pf.: $B([s_\alpha, s_\beta]u, v) = B(u, [s_\beta, s_\alpha]v) \Rightarrow [s_\alpha, s_\beta] \in \mathfrak{so}(V, B) \cong \wedge^2(V)$, which is a W-irrep. □
- If $\varphi : V^* \cong V$ with $\lambda(v) = B(\varphi(\lambda), v)$, we have

$$\nu([s_\alpha, s_\beta]) = B(\alpha^\vee, \beta^\vee)(\varphi(\alpha)\beta - \varphi(\beta)\alpha) \in \mathcal{W}.$$

- Define $D^+ = \sum_i \widetilde{v_i} \otimes v^i$ and $D^- = \sum_i \widetilde{v_i} \otimes v_i^*$.
- **Thm**(C-DM-M): In $\mathbb{H} \otimes \mathcal{W}$ we have

$$[D^+, D^-] = -(\Omega_{\mathbb{H}} - \Omega_W - \Omega'_W) \otimes 1 + 1 \otimes \nu(\Omega'_W) - \Delta_\nu(\Omega'_W)$$

$$\Omega_{\mathbb{H}} = \sum_i v_i v^i \in Z(\mathbb{H}),$$

$$\Omega_W = \tfrac{1}{4} \sum_{\alpha,\beta>0} c_\alpha c_\beta B(\alpha, \beta) s_\alpha s_\beta \in Z(\mathbb{C}[W]),$$

$$\Omega'_W = \tfrac{1}{16} \sum_{\alpha,\beta>0} c_\alpha c_\beta B(\alpha^\vee, \beta^\vee)^{-1} [s_\alpha, s_\beta]^2 \in Z(\mathbb{C}[W]).$$

- The elements $\Omega_{\mathbb{H}}, \Omega_W$ appear in the orthogonal case.
- **Thm**(C-DM-M): Ω'_W satisfies the following properties:
  - $\nu(\Omega'_W) = a + b\Omega(\mathfrak{sl}(2))$, with a, b propotional to $(\nu(\Omega'_W))_0$.
  - $(\nu(\Omega'_W))_0 = \frac{1}{2}B(\rho_k, \rho_k) - \frac{1}{8}\sum_{\alpha,\beta>0} k(\alpha)k(\beta)\frac{B(\alpha,\beta)^3}{B(\alpha,\alpha)B(\beta,\beta)}$.

  Pf.: Write $X = X_{\alpha\beta} = m(\nu([s_\alpha, s_\beta]))$ and $\Omega'_W = \sum_{\alpha,\beta} c_{\alpha\beta} X^2_{\alpha\beta}$. Let $\xi = v_1 \cdots v_p \in S^p(V)$ then

$$X^2(\xi) = \sum_i \cdots X^2(v_i) \cdots + \sum_{i \neq j} \cdots X(v_i)X(v_j) \cdots$$

  but $\sum_{\alpha,\beta} c_{\alpha\beta} X_{\alpha\beta}(v_i v_j) = 0$. Hence, for all $Y \in \nu(\mathfrak{so}(V, B))$

$$m(Y)m(\Omega'_W)(\xi) = \sum_i \cdots m(Y)m(\Omega'_W)(v_i) \cdots = m(\Omega'_W)m(Y)(\xi)$$

  and hence $\nu(\Omega'_W)$ is $O(V, B)$-invariant. □

§4 Further results.

- Computation of $\ker(D^\pm)$ in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.
- Applications to unitarity:
  - LA, extending $x^* = -\bar{x}$ to $\mathcal{U}(\mathfrak{g})$:

$$\chi_M(\Omega(\mathfrak{g})) \leq \sigma(\Omega(\mathfrak{k})), \text{ for } M \in \mathrm{Irr}(\mathfrak{g}), \sigma \in \mathrm{Irr}(\mathfrak{k}), M(\sigma) \neq 0.$$

  - GAHA, extending $w^* = w^{-1}, \widetilde{v}^* = -\widetilde{v}$ to $\mathbb{H}$:

$$B(\chi_M, \chi_M) \leq \sigma(\Omega_W), \text{ for } \in \mathrm{Irr}(\mathbb{H}), \sigma \in \mathrm{Irr}(W), M(\sigma) \neq 0.$$