Unramified Spherical Automorphic Spectrum: Computational Aspects

Celebration of E. Opdam's 60^{th}*-Birthday*

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Outline

The Problem

- G: split, almost simple LAG over a number field F.
- $K = \prod_v K_v$: max'l compact, with $K_v = G(\mathfrak{o}_v)$ for all v finite.
- $[G] = G(F)\backslash G(\mathbb{A})$: automorphic quotient.

Theorem (Langlands)

Decomposition into cuspidal data (M Levi, π cuspidal automorphic of M)

$$L^2([G]) = \hat{\bigoplus}_{[M,O(\pi)]} L^2([G])_{[M,O(\pi)]}$$

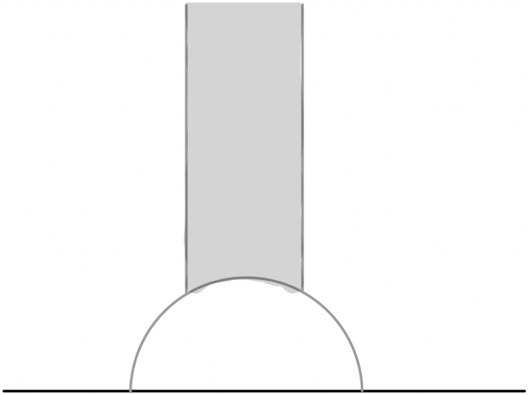
- $B \supset T$: Borel subgroup, maximal split torus ($\rightsquigarrow W, R, R^\vee, R_+, R_+^\vee$).

Problem

Decompose $L^2([G])^K_{[T,1]}$ as a module for $\mathcal{H} = \otimes'_v \mathcal{H}(G_v, K_v)$.

For $F = \mathbb{Q}$, $G = GL_2$, $K = SO_2(\mathbb{R}) \times \prod_p GL(\mathbb{Z}_p)$, then

$$Z(\mathbb{A})GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K \cong SL_2(\mathbb{Z})\backslash\mathbb{H}.$$



The Starting Point

- Let $m_B : G(\mathbb{A}) = B(\mathbb{A})K \to T(\mathbb{A})/T^1 \cong X_*(T) \otimes \mathbb{R}_+$.

Definition (Eisenstein Series)

For $g \in G(\mathbb{A})$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^* := X^*(T) \otimes \mathbb{C}$

$$\mathcal{E}(\lambda, g) = \sum_{\gamma \in B(F)\backslash G(F)} m_B(\gamma g)^{\lambda+\varrho}$$

Theorem (Properties of the Eisenstein Series)

- absolutely convergent if $\mathrm{Re}(\lambda - \varrho) \succ 0$.
- meromorphic continuation to $\mathfrak{a}_{\mathbb{C}}^*$.
- $\mathcal{E}(\lambda, \cdot)$ is an $\mathcal{H}$-eigenform with eigenvalue χ_λ (Satake).

Definition (Pseudo-Eisenstein Series)

Fix $\lambda_0 \in \mathfrak{a}^*$ with $(\lambda_0 - \varrho) \succ 0$ and $\phi \in PW(\mathfrak{a}^*_{\mathbb{C}})$

$$\theta_\phi(g) := \int_{\mathrm{Re}(\lambda)=\lambda_0} \phi(\lambda)\mathcal{E}(\lambda, g) d\lambda.$$

Theorem (Langlands)

$$L^2([G])^K_{[T,1]} = \overline{\mathrm{span}(\{\theta_\phi \mid \phi \in PW(\mathfrak{a}^*_{\mathbb{C}})\})},$$

$$(\theta_\phi, \theta_\psi) = \int_{\mathrm{Re}(\lambda)=\lambda_0} \sum_{w \in W} \prod_{\alpha \in R(w)} \frac{\Lambda_F(\langle \lambda, \alpha^\vee \rangle)}{\Lambda_F(\langle \lambda, \alpha^\vee \rangle + 1)} \phi^-(-w\lambda)\psi(\lambda) d\lambda,$$

where $\phi^-(\lambda) = \overline{\phi(\overline{\lambda})}$ and Λ_F is the completed Dedekind zeta function.

Historical Notes

- (Moeglin '90): conjecture that only poles of Λ_F contribute to spectrum.
- Langlands ('76), Jacquet ('84), Moeglin-Waldspurger ('89), Moeglin ('91), Kim ('96, '01) and others.
- Recent breakthrough: Kazhdan-Okounkov ('22).

Outline

General Setup

- Heckmann-Opdam (‘97), Opdam (‘04, ‘07).
- Let $V \cong \mathbb{R}^n$ oriented Euclidean space $V_{\mathbb{C}} = V \otimes \mathbb{C}$.
- Let $\mathcal{A}$ be a finite set of affine hyperplanes of V, $\mathcal{A}_{\mathbb{C}}$ the set of complexified hyperplanes.
- Let ω an $(n, 0)$-form on $V_{\mathbb{C}}$ with singular and zero locus in $\mathcal{A}_{\mathbb{C}}$.
- For $p \in V \setminus (\cup_{H \in \mathcal{A}} H)$, let $\mathcal{X}^{\omega,p} : PW(V_{\mathbb{C}}) \to \mathbb{C}$

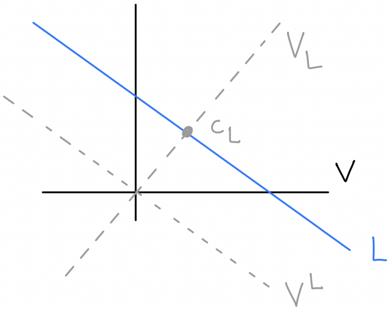
$$\mathcal{X}^{\omega,p}(\phi) = \int_{\mathrm{Re}(\lambda)=p} \phi(\lambda)\omega(\lambda)$$

- Denote $L(\mathcal{A})$ the intersection semilatice of $\mathcal{A}$.

- $L \subset V$ affine space, let V^L be the vector part, $V_L = (V^L)^\perp$
- Let c_L the center of L characterized by $\{c_L\} = V_L \cap V^L$. Have

$$L = c_L + V^L \subseteq V$$

- $L^{temp} = c_L + iV^L \subset V_{\mathbb{C}}$.



Definition

For $H \in \mathcal{A}$ and $L \in L(\mathcal{A})$ let

- $n_H = \omega$-order of H ($n_H \in \mathbb{Z}$),
- $n_L = \sum_{H \supset L} n_H$,
- L is said ω-residual if $L = \cap\{H \mid H \supset L \text{ and } n_H < 0\}$ and

$$o_L^\omega := n_L + \mathrm{codim}(L) \leq 0.$$

- $\mathcal{C}^\omega = \{c_L \mid L \in L(\mathcal{A}) \text{ is } \omega\text{-residual}\}$.

Theorem (Heckman-Opdam)

Exists unique collection of tempered distributions $\{\mathcal{X}_c \mid c \in \mathcal{C}^\omega\}$ with

$$\mathrm{Supp}(\mathcal{X}_c) \subseteq \cup_{L\ \omega\text{-residual: } c_L = c} L^{temp}$$

$$\mathcal{X}^{\omega,p}(\phi) = \sum_{c \in \mathcal{C}^\omega} \mathcal{X}_c(\phi|_{c+iV})$$

Back to the Spherical Automorphic Spectrum

- Back to the initial setup. Let:

$$c(\lambda) := \prod_{\alpha \in R_+} \frac{\langle \lambda, \alpha^\vee \rangle + 1}{\langle \lambda, \alpha^\vee \rangle}$$

$$\rho(s) := s(1-s)\Lambda_F(s)$$

$$r(\lambda) := \prod_{\alpha \in R_+} \rho(\langle \lambda, \alpha^\vee \rangle)$$

$$(\Sigma_W(\psi))(\lambda) := \sum_{w \in W} c(-w\lambda)\psi(-w\lambda)\frac{r(\lambda)}{r(w\lambda)} \quad (\psi \in PW(\mathfrak{a}^*)).$$

Proposition (DM, Heiermann, Opdam)

$$(\theta_\phi, \theta_\psi) = \int_{\mathrm{Re}(\lambda)=\lambda_0} (\Sigma_W(\phi^-)\ \psi)(\lambda) \frac{d\lambda}{c(-\lambda)}.$$

Algebraic Poles

- Let $X^{\lambda_0}, Y^{\lambda_0} : PW(\mathfrak{a}^*_{\mathbb{C}}) \to \mathbb{C}$ with

$$X^{\lambda_0}(\phi) = \int_{\mathrm{Re}(\lambda)=\lambda_0} \phi(\lambda)\omega^X(\lambda) \qquad \omega^X(\lambda) = \tfrac{d\lambda}{c(-\lambda)},$$
$$Y^{\lambda_0}(\phi) = \int_{\mathrm{Re}(\lambda)=\lambda_0} \phi(\lambda)\omega^Y(\lambda) \qquad \omega^Y(\lambda) = \tfrac{d\lambda}{c(\lambda)c(-\lambda)}.$$

- Note that $\frac{1}{|W|}\sum_{w\in W} \frac{1}{c(-w\lambda)} = \frac{1}{c(\lambda)c(-\lambda)}$.
- If $p \in \mathfrak{a}^*$ let $A_p(\phi)(\lambda) = \frac{1}{|W_p|}\sum_{w\in W_p} c(w\lambda)\phi(w\lambda)$.

Theorem (Heckman-Opdam)

For $w \in W$ and $c \in \mathfrak{a}_+^*$ (the dominant chamber)

$$X_{wc}(\phi|_{wc+i\mathfrak{a}^*}) = Y_c((A_{wc}(\phi) \circ w)|_{c+i\mathfrak{a}^*})$$

Theorem (Heckman-Opdam)

For any affine subspace $L \subset \mathfrak{a}^*$, L is ω^Y-residual iff $o_L^Y = 0$:

$$|\{\alpha \in R \mid \alpha^\vee|_L = 1\}| = |\{\alpha \in R \mid \alpha^\vee|_L = 0\}| + \mathrm{codim}(L).$$

- Say "residual $= \omega^Y$-residual". Let $\mathcal{L}$ be the set of all residual spaces.

Theorem (Opdam)

$$\begin{array}{rcl} W\backslash\mathcal{L} & \stackrel{\cong}{\to} & \{\text{WDD of nilpotent orbits of } \mathfrak{g}^\vee\} \\ WL & \mapsto & 2c_{L,+}. \end{array}$$

Normalised Eisenstein Series

Definition (Normalised Eisenstein and Pseudo-Eisentein Series)

$$\mathcal{E}_0(\lambda, g) := |W|^{-1} r(-\lambda) c(-\lambda) \mathcal{E}(\lambda, g) \qquad (\lambda \in \mathfrak{a}^*_{\mathbb{C}}, g \in G(\mathbb{A}))$$

$$\theta_{\phi,0} := X^{\lambda_0}(\phi \mathcal{E}_0(\bullet, g)) \qquad (\phi \in PW(\mathfrak{a}^*_{\mathbb{C}}))$$

Proposition (DM, Heiermann, Opdam)

- $\mathcal{E}_0(\lambda, g)$ is entire, W-invariant and $\theta_{\phi,0} \in L^2([G])^K_{[T,1]}$.
- $(\theta_{\phi,0}, \theta_{\psi,0}) = \frac{1}{|W|} X^{\lambda_0}(\ r(\bullet) r(-\bullet) \psi(\bullet) A_0(\phi^-)(-\bullet)\ )$

- Let $L^2([G])^K_{[T,1],0} = \overline{\mathrm{span}(\{\theta_{\phi,0} \mid \phi \in PW(\mathfrak{a}^*_{\mathbb{C}})\})}$.
- Let $\Xi := W \backslash \cup_{L \in \mathcal{L}} L^{temp}$.

Upshot (DM, Heiermann, Opdam)

Have

$$L^2([G])^K_{[T,1]} = L^2([G])^K_{[T,1],0} \oplus \text{(orthogonal complement)}$$

Can describe $L^2([G])^K_{[T,1],0} \cong L^2(\Xi, \mu_0)$ in terms of nilpotent orbits $\mathfrak{g}^\vee$ and for an explictly known positive measure μ_0, smooth on each component.

- Description of $L^2([G])^K_{[T,1],0}$ generalises for G quasi-split and more general χ.

Outline

Summary so far

$$\begin{pmatrix} \text{decomposition of} \\ L^2([G])^K_{[T,1]} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{evaluation of} \\ (\theta_\phi, \theta_\psi) \end{pmatrix}$$

$$\begin{pmatrix} \text{evaluation of} \\ (\theta_\phi, \theta_\psi) \end{pmatrix} \longleftrightarrow \begin{pmatrix} \int_{\mathrm{Re}(\lambda)=\lambda_0} \bullet \frac{d\lambda}{c(-\lambda)} + \\ (\Sigma_W(\phi^-)\,\psi)(\lambda) \end{pmatrix}$$

$$\begin{pmatrix} \int_{\mathrm{Re}(\lambda)=\lambda_0} \bullet \frac{d\lambda}{c(-\lambda)} + \\ \text{normalised Eisenstein} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{expected spectrum} \\ L^2([G])^K_{[T,1],0} \end{pmatrix}$$

- Need extra combinatorial tool to conclude

$$(\text{orthogonal complement}) = 0$$

- Computer-assisted computations, using Maple™.

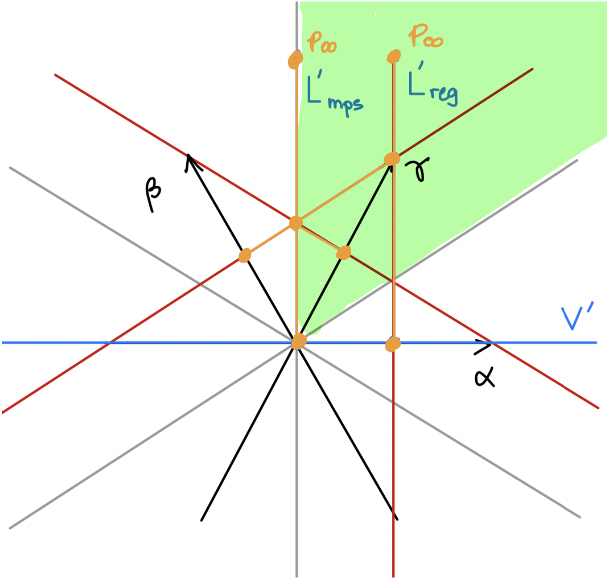
Contour-shift à la Moeglin

- Choose proper Levi $G' \subset G$.
- Assume the expected result is true for G' (inner product is given in terms of residual spaces for G').
- Focus on the pairs $(R'^{\vee}, R^{\vee})$ with:
  - (X_{n-1}, X_n) for X classical,
  - (D_5, E_6), (E_6, E_7) and (E_7, E_8),
  - (C_3, F_4),
  - (A_1, G_2).

- If $L' \in \mathcal{L}'_+$ let $L = L' \oplus \mathbb{R}\varpi'$ and $p_{L,\infty} = c'_L + t\varpi'$ with $t \gg 0$.
- Put $\Omega(\lambda) = \frac{d\lambda}{c'(\lambda)c(-\lambda)}$ and let $\Omega^L(\lambda)$ be the residue along L.
- By induction we can assume

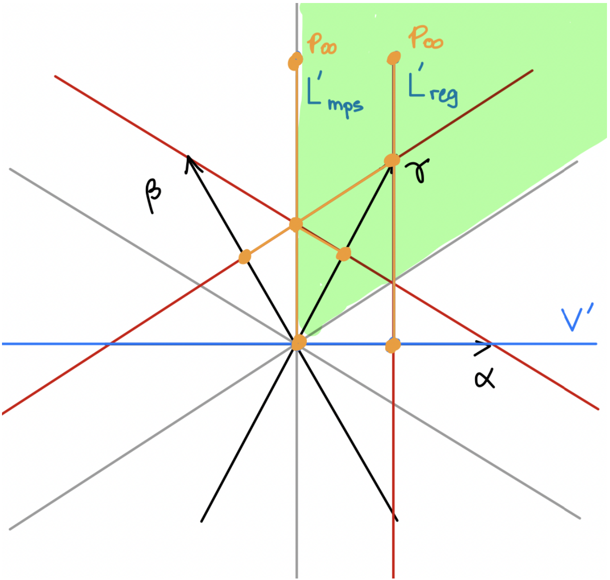
$$(\theta_\phi, \theta_\psi) \text{``} = \text{''} \sum_{L' \in \mathcal{L}'_+} \int_{p_{L,\infty} + i(\mathfrak{a}^*)^L} \Sigma_{W'}(\psi)\Sigma_W(\phi^-)\Omega^L(\lambda)$$

$(\Sigma_{W'}(\psi))(\lambda) = \sum_{u \in W'} c(u\lambda)\psi(u\lambda)\frac{r(u\lambda)}{r(\lambda)}$.


P_∞
P_∞
L'_{mps}
L'_{reg}
β
γ
V'
α

A. The Cascade of Contour Shifts

- Want to classify all spaces crossed when symmetrizing $(\theta_\psi, \theta_\phi)$.
- We organize the data in a combinatorial object: *cascade* $(C, Std(C))$
  - C: a set of pairs (L, σ), with $\sigma = [i_\sigma, f_\sigma] \subset L$.
  - $Std(C)$: is a set of pairs (L_0, w) with $L = wL_0 \in C$ stored up to W'-conjugation.
- Algorithm:
  - Input: WDD of all nipotent orbits of $\mathfrak{g}'^\vee$.
  - 1^{st} step: for each L', shift $[p_{L,\infty}, c_L]$ and cross poles M of Ω^L at a point p_M.
  - *main decision:* if exists $N \in \mathcal{L}$, $M \subseteq N$ and $c_M = c_N$: shift $[p_M, c_M]$.
  - else: shift $[p_M, p'_M]$ to some other initial point $p'_M \in M$.
  - in each step, store std representative (M_0, w) up to W'-conjugation.
  - store the *order* $\mathrm{Ord}_L(\Omega)$ of each $L \in C$.


P_∞
P_∞
L'_{mps}
L'_{reg}
β
γ
V'
α

B. Enveloping Denominators

- Let $L = wL_0$, with $\text{Ord}_L(\Omega) = 0$ and passing the *main decision*.
- Let $Den(L)$ be the denominators of $(\Sigma_W(\phi^-)\Sigma_{W'}(\psi))|_L$.

$$Den(L) = (w^{-1})^*(Den(L_0, w, \Sigma)) \cup (w^{-1})^*(Den(L_0, w, \Sigma'))$$

- If L singular, it is hard to determine $Den(L)$.
- For L *regular* we can control things better.
- Have $Den(L_0, w, \Sigma/\Sigma') \subset \cap_{H \supset L_0} Den(H, w, \Sigma/\Sigma')$ with H regular.
- Algorithm:
  - Given L_0 find all* regular envelopes $H \supset L_0$
  - Compute $Den(H, w, \Sigma/\Sigma')$ and intersect.
  - Check if $\sigma = [i_\sigma, f_\sigma]$ is in the same side of the critical strip for each approximated denominator.

C. Higher order spaces

- For almost all $(L, \sigma) \in C$ with $\mathrm{Ord}_L(\Omega) > 0$ we have
  - either $\sigma = [p, p]$ or
  - for all $M \supset L$ have σ is in the same side of all critical strips of approximated denominators of M.
- There is *one* exception (L_{sp}, σ_{sp}) in (E_7, E_8) for L_{sp} of type $E_7(a_4)$.
- There is a unique troublesome flag $N \supset L_{sp}$:

$$N \leftrightarrow \left(\begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & x & y \\ & & 0 & & & & \end{smallmatrix} \right)$$

$$L_{sp} \leftrightarrow \left(\begin{smallmatrix} 1 & 0 & 1 & 0 & 1 & x & -1-x \\ & & 0 & & & & \end{smallmatrix} \right) = s_8 s_7 \left(\begin{smallmatrix} 1 & 0 & 1 & 0 & 0 & 1 & x \\ & & 0 & & & & \end{smallmatrix} \right).$$

- There is a unique approximated denominator of N separating σ_{sp}.
- Classify all $w \in W_8$ with $w = \eta\tau u$ $(u \in W_6, \tau \in W_{7/6}, \eta \in W_{8/7})$

$$|P_N \cap R^\vee(u)| \leq 3, |P_{L_{sp}} \cap R^\vee(w)| \leq 6.$$

- Compute $\partial_{\langle h, P(w) \rangle} S(-w)|_{L_{sp,0}}$ for certain h harmonic polys (multiplied by a normal direction) and check for vanishing of coefficients.

Thank You!
Congratulations, Eric.

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https://github.com/mgdemartino/SphAutSpec