A Quantized Metaplectic Howe Duality in Rank One

(joint with M. Brito)

Marcelo De Martino

Forward College, Lisbon

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Outline

Motivation: Dirac Operators

- **The "Dirac plot":**

$$\text{operators} \longrightarrow \text{special functions} \longrightarrow \text{symmetry algebra.}$$

- **The construction:**

$$(V,B) \rightsquigarrow \mathcal{C}(V,B) \rightsquigarrow \mathcal{W}(V)\otimes\mathcal{C}(V,B) \curvearrowright \mathcal{P}(V)\otimes\mathcal{S} \rightsquigarrow D \in \mathcal{W}(V)\otimes\mathcal{C}(V,B).$$

Motivation: Dualities

- **Howe duality viewpoint:** $(\mathfrak{g}, \mathfrak{g}')$ duality

$$U(\mathfrak{g}) \curvearrowright \mathcal{M} \curvearrowleft U(\mathfrak{g}') \quad \rightsquigarrow \quad \mathcal{M} = \bigoplus_{\lambda} L_{\mathfrak{g}}(\lambda) \otimes L_{\mathfrak{g}'}(\lambda')$$

Take-home goal (today)

Construct an *explicit* quantization of the metaplectic Howe duality that contains the **symplectic Dirac operator** $(\mathfrak{sp}_{2n}, \mathfrak{sl}_2)$ for the first nontrivial case, $n = 1$:

$$\big(U_{q^2}(\mathfrak{sl}_2),\ U_q(\mathfrak{sl}_2)\big).$$

Outline

Orthogonal world

- **Clifford algebra:** $\mathcal{C}$, with generators $(e_1, \ldots, e_n)$ with
$$e_i e_j + e_j e_i = (e_i, e_j),$$
$\mathcal{S} = \Lambda(\mathbb{C}^{\lfloor n/2 \rfloor})$.
- **Orthogonal Dirac operator:**
$$D := \sum_{j=1}^{n} \partial_{e_j} \otimes e_j^* \in \mathcal{W}_n \otimes \mathcal{C} \qquad \text{(with adjoint operator } X\text{)}$$
- **Two classical dualities to keep in mind:**
$$\begin{array}{ccccc} \Delta, \ r^2 & \rightsquigarrow & \mathfrak{sl}_2 & \rightsquigarrow & U(\mathfrak{so}_n) \curvearrowright \mathcal{P} \curvearrowleft U(\mathfrak{sl}_2) \\ D, \ X & \rightsquigarrow & \mathfrak{spo}(2|1) & \rightsquigarrow & U(\mathfrak{so}_n) \curvearrowright \mathcal{P}(\mathcal{S}) \curvearrowleft U(\mathfrak{spo}(2|1)) \end{array}$$

Orthogonal world: Fischer Decompositions, pt 1

$$
\begin{tikzcd}
\mathcal{P}^0 \arrow[d, phantom, "\parallel"] & \mathcal{P}^1 \arrow[d, phantom, "\parallel"] & \mathcal{P}^2 \arrow[d, phantom, "\parallel"] & \mathcal{P}^3 \arrow[d, phantom, "\parallel"] & \cdots \\
\mathcal{H}^0 \arrow[rr, "r^2"] & & r^2(\mathcal{H}^0) \arrow[rr, "r^2"] & & \cdots \\
& \mathcal{H}^1 \arrow[rr, "r^2"] & & r^2(\mathcal{H}^1) \arrow[r, "r^2"] & \cdots \\
& & \mathcal{H}^2 \arrow[rr, "r^2"] & & \cdots \\
& & & \mathcal{H}^3 \arrow[r, "r^2"] & \cdots
\end{tikzcd}
$$

Orthogonal world: Fischer Decompositions, pt 2

$$
\begin{tikzcd}
\mathcal{P}(\mathcal{S})^0 \arrow[d, phantom, "\parallel"] & \mathcal{P}(\mathcal{S})^1 \arrow[d, phantom, "\parallel"] & \mathcal{P}(\mathcal{S})^2 \arrow[d, phantom, "\parallel"] & \mathcal{P}(\mathcal{S})^3 \arrow[d, phantom, "\parallel"] & \cdots \\
\mathcal{M}^0 \arrow[r, "X"] & X(\mathcal{M}^0) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & X^2(\mathcal{M}^0) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & X^3(\mathcal{M}^0) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & \cdots \\
 & \mathcal{M}^1 \arrow[r, "X"] & X(\mathcal{M}^1) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & X^2(\mathcal{M}^1) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & \cdots \\
 & & \mathcal{M}^2 \arrow[r, "X"] & X(\mathcal{M}^2) \arrow[r, "X"] \arrow[d, phantom, "\oplus"] & \cdots \\
 & & & \mathcal{M}^3 \arrow[r, "X"] & \cdots
\end{tikzcd}
$$

Symplectic world

- **Symplectic Clifford algebra:** $\mathcal{C}_s$, with generators $(e_1, \ldots, e_{2n})$ with
$$e_i e_j - e_j e_i = \omega(e_i, e_j),$$
$\mathcal{S} = \mathcal{P}(\mathbb{C}^n)$.
- **Key structural difference:** *two* natural symplectic Dirac-type operators
$$D_s = \sum_{j=1}^{2n} \partial_{e_j} \otimes e_j^* \in \mathcal{W}(2n) \otimes \mathcal{C}_s \qquad \text{(with adjoint operator } X_s)$$
and $\widetilde{D}_s = J D_s$.
- **Metaplectic Howe duality:** [De Bie, Soucek, Stromberg, '14]
$$D_s, \quad X_s \quad \rightsquigarrow \quad U(\mathfrak{sp}_{2n}) \curvearrowright \mathcal{P}_{2n} \otimes \mathcal{S} \curvearrowleft U(\mathfrak{sl}_2)$$

Outline

Quantization of Howe dualities

- **Expectation:**
  - replace enveloping algebras by "quantum groups"
  - commutation/centralizer behavior
  - clean decomposition patterns
- **Examples:**
  - [Zhang, '03]: $U_q(\mathfrak{gl}_m) \curvearrowright S_q(\mathbb{C}^m \otimes \mathbb{C}^n) \curvearrowleft U_q(\mathfrak{gl}_n)$
  - [Futorny, Krizka, Zhang, '19]: $U_q(\mathfrak{sl}_n) \curvearrowright S_q(\mathbb{C}^n) \curvearrowleft U_q(\mathfrak{sl}_2)$
  - [Zhang, '20]: $U_q(\mathfrak{gl}_{k|l}) \curvearrowright S_q(\mathbb{C}^{k|l} \otimes \mathbb{C}^{r|s}) \curvearrowleft U_q(\mathfrak{gl}_{r|s})$
  - [Bodish, Tubenhauer, '25]: $U_q(\mathfrak{sp}_{2n}) \curvearrowright \Lambda_q(\mathbb{C}^{2n}) \curvearrowleft U_q(\mathfrak{sl}_2)$
  - ...

Orthogonal Cases: Harmonic Pair

- [Noumi, Umeda, Wakayama, '96] and [Iorgov, Klimyck, '01]
- Fix $q \neq \{\pm 1, \pm i\}$, let $\mathbb{K} = \mathbb{Q}(q)$ and denote $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + \cdots + q^{1-m}$
- Define $\gamma, \partial_q \in \mathrm{End}(\mathbb{K}[x])$ via

$$\gamma(x^m) = q^m x^m, \quad \partial_q = \frac{1}{x} \frac{\gamma - \gamma^{-1}}{q - q^{-1}}$$

- $\pi : U_{q^2}(\mathfrak{sl}_2) \to \mathrm{End}(\mathbb{K}[x])$ via

$$\pi(e) = \frac{x^2}{[2]_q}, \quad \pi(f) = -\frac{\partial^2}{[2]_q}, \quad \pi(k) = q\gamma^2$$

- Note: $\partial_q x^m = [m]_q x^{m-1}$

Orthogonal Cases: Harmonic Pair

- Using $\Delta(e) = e \otimes 1 + k^{-1} \otimes e, \quad \Delta(e) = f \otimes k + 1 \otimes f, \quad \Delta(k) = k \otimes k$

$$\pi : U_q(\mathfrak{sl}_2) \to \mathrm{End}(\mathbb{K}[x_1, x_2, \ldots, x_n])$$

- Inside $\mathrm{End}(\mathbb{K}[x_1, x_2, \ldots, x_n])$ the operators commuting with $\pi(U_q(\mathfrak{sl}_2))$ are:

$$U_q'(\mathfrak{o}_n) \twoheadrightarrow U_q \subset \mathrm{End}(\mathbb{K}[x_1, x_2, \ldots, x_n])$$

with $U_q'(\mathfrak{o}_n)$ generated by $\Theta_1, \ldots, \Theta_{n-1}$ subject to

$$\begin{cases} [\Theta_j, \Theta_k] = 0 & \text{if } |j-k| > 1 \\ \Theta_j^2\Theta_k - (q+q^{-1})\Theta_j\Theta_k\Theta_j + \Theta_k\Theta_j^2 = -\Theta_k & \text{if } |j-k| = 1. \end{cases}$$

Orthogonal Cases: Two q-Dirac operators, two theories

- [Coulembier, Sommen, '10]: they define D_q (in Clifford Analysis, $\partial_{\underline{x}}^q$) via

$$\text{(A1) } D_q(\underline{x}) = [m]_q \qquad \text{(A2) } D_q\underline{x}^2 = q^2\underline{x}^2 D_q + (q+1)\underline{x}$$
$$\text{(A3) } D_q^2 \text{ is scalar} \qquad \text{(A4) } D_q(M_k) = 0$$

- (A1) – (A4) uniquely define $D_q \rightsquigarrow U_q(\mathfrak{spo}(2|1))$, $U_q(\mathfrak{sl}_2)$.
- [Bernstein, Schneider, Zimmermann '22]:

$$D_q = \sum_j \partial_{j,q} e_j$$

- Axioms (A1), (A2) are not satisfied, lead a Fischer decomposition, but no quantum groups.

Outline

Hayashi's deformed Weyl algebra

Definition (Hayashi's q-Weyl algebra)

Fix $q \in \mathbb{C}^\times \setminus \{\pm 1\}$. $\mathcal{W}_{N,q}$ is the unital associative $\mathbb{C}$-algebra generated by

$$\mu_j,\ \partial_j,\ \gamma_j^{\pm 1} \qquad (1 \leq j \leq N),$$

subject to

$$\gamma_j \gamma_k = \gamma_k \gamma_j, \qquad \gamma_j \gamma_j^{-1} = 1,$$

$$\gamma_j \partial_k \gamma_j^{-1} = q^{-\delta_{jk}} \partial_k, \qquad \gamma_j \mu_k \gamma_j^{-1} = q^{\delta_{jk}} \mu_k,$$

$$[\partial_j, \partial_k] = 0 = [\mu_j, \mu_k], \qquad [\partial_j, \mu_k] = 0 \ \ (j \neq k),$$

$$\partial_j \mu_j - q^{\pm 1} \mu_j \partial_j = \gamma_j^{\mp 1}.$$

Action on polynomials

- Have $\mathcal{W}_{N,q} \curvearrowright \mathcal{P}_N = \mathbb{C}[x_1, \cdots, x_N] \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$

$$\begin{aligned} \mu_j(x^\alpha) &= x^{\alpha+e_j} \\ \partial_j(x^\alpha) &= [\alpha_j]_q x^{\alpha-e_j} \\ \gamma_j^{\pm}(x^\alpha) &= q^{\pm\alpha_j} x^\alpha \end{aligned}$$

- $\partial_{q^m} = \frac{1}{x}\frac{\gamma^m - \gamma^{-m}}{q^m - q^{-m}} = \partial_q \frac{[m]_\gamma}{[m]_q}, m = 1, 2, 3, \ldots$
- Recall $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + \cdots + q^{1-m}$

Realizations in rank one, pt. 1

- Aim: $(\mathfrak{sp}_2, \mathfrak{sl}_2) \subset \mathcal{W}_2 \otimes \mathcal{C}_s$, $(\mathfrak{sp}_2, \mathfrak{sl}_2) \curvearrowright \mathbb{C}[x_1, x_2] \otimes \mathbb{C}[y]$ and q-version
- Realizations in the Weyl part: [Zhang-Hu, '17]

$\mathfrak{sp}_2$ in $\mathcal{W}_2$	$U_{q^2}(\mathfrak{sl}_2) \twoheadrightarrow \mathcal{W}_{2,q}$
$E_w = \mu_1 \partial_2,$	$E_w = \mu_1 \partial_{2,q^2}$
$F_w = \mu_2 \partial_1,$	$F_w = \mu_2 \partial_{1,q^2}$
$H_w = \mu_1 \partial_1 - \mu_2 \partial_2$	$K_w := \gamma_1^2 \gamma_2^{-2}$

- Realizations in the Clifford part (here $\mathcal{C}_{s,q} = \mathbb{C}\langle \nabla, \nu, \eta^{\pm 1} \rangle$): [Hayashi, '90]

$\mathfrak{sp}_2$ in $\mathcal{C}_s$	$U_{q^2}(\mathfrak{sl}_2) \twoheadrightarrow \mathcal{C}_{s,q}$
$E_c = \frac{1}{2}\nu^2,$	$E_c := \frac{1}{[2]_q}\nu^2$
$F_c = -\frac{1}{2}\nabla^2$	$F_c := -\frac{1}{[2]_q}\nabla^2$
$H_c = \nu\nabla + \frac{1}{2}$	$K_c := q\eta^2$

Realizations in rank one, pt. 2

- Diagonal Realization:

$\mathfrak{sp}_2$ in $\mathcal{W}_2 \otimes \mathcal{C}_s$	$U_{q^2}(\mathfrak{sl}_2) \twoheadrightarrow \mathcal{W}_{2,q} \otimes \mathcal{C}_{s,q}$
$E_\Delta = E_w \otimes 1 + 1 \otimes E_c$	$E_\Delta = E_w \otimes K_c + 1 \otimes E_c$
$F_\Delta = F_w \otimes 1 + 1 \otimes F_c$	$F_\Delta = F_w \otimes 1 + K_w^{-1} \otimes F_c$
$H_\Delta = H_w \otimes 1 + 1 \otimes H_c$	$K_\Delta = K_w \otimes K_c$

- Dual algebra: [Brito - DM, '25]

$\mathfrak{sl}_2$ in $\mathcal{W}_2 \otimes \mathcal{C}_s$	$U_q(\mathfrak{sl}_2) \twoheadrightarrow \mathcal{W}_{2,q} \otimes \mathcal{C}_{s,q}$
$E = 2(\mu_1 \nabla + \mu_2 \nu)$	$E = [2]_q(q^2\gamma_2^2\eta\mu_1\nabla + \mu_2\nu)$
$F = \partial_1 \nu - \partial_2 \nabla$	$F = \partial_{1,q^2}\nu - \gamma_1^{-2}\eta\partial_{2,q^2}\nabla$
$H = 2(\mu_1\partial_1 + \mu_2\partial_2 + 1)$	$K = q^2\gamma_1^2\gamma_2^2$

The Fischer Decomposition

Thm (Brito, DM)

(a) The d-monogenics $\mathcal{M}^d := \ker(F) \cap \mathcal{P}^d(\mathcal{S})$ decompose as $\mathcal{M}^d = \mathcal{M}^d_+ \oplus \mathcal{M}^d_-$

(b) As $U_{q^2}(\mathfrak{sl}_2)$-modules:

$$\mathcal{M}^d_+ \cong L_{q^2}\left(\tfrac{1}{2} - d\right), \qquad \text{generated by } \mathbf{p}_+ = x_2^d \otimes 1$$

$$\mathcal{M}^d_- \cong L_{q^2}\left(\tfrac{3}{2} + d\right), \qquad \text{generated by } \mathbf{p}_- = \sum_{b=0}^{d} \begin{bmatrix} d \\ b \end{bmatrix}_{q^2} \frac{q^{2b(d-b-1)}}{[2b+1]_q!!} x_1^{d-b} x_2^b \otimes y^{2b+1}$$

(c) For q **not a root of unity**, joint action $\left(U_{q^2}(\mathfrak{sl}_2), U_q(\mathfrak{sl}_2)\right)$ decomposes $\mathcal{P}(\mathcal{S})$ as

$$\mathcal{P}(\mathcal{S}) = \bigoplus_{d \geq 0} \left(L_{q^2}(\tfrac{1}{2} - d) \oplus L_{q^2}(\tfrac{3}{2} + d)\right) \otimes L_q(2d+2).$$

The Fischer Triangle

$$
\begin{array}{ccccccccc}
\mathcal{P}(\mathcal{S})^0 & & \mathcal{P}(\mathcal{S})^1 & & \mathcal{P}(\mathcal{S})^2 & & \mathcal{P}(\mathcal{S})^3 & & \cdots \\
\| & & \| & & \| & & \| & & \\
\mathcal{M}^0 & \xrightarrow{E} & E(\mathcal{M}^0) & \xrightarrow{E} & E^2(\mathcal{M}^0) & \xrightarrow{E} & E^3(\mathcal{M}^0) & \xrightarrow{E} & \cdots \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & \mathcal{M}^1 & \xrightarrow{E} & E(\mathcal{M}^1) & \xrightarrow{E} & E^2(\mathcal{M}^1) & \xrightarrow{E} & \cdots \\
 & & & & \oplus & & \oplus & & \\
 & & & & \mathcal{M}^2 & \xrightarrow{E} & E(\mathcal{M}^2) & \xrightarrow{E} & \cdots \\
 & & & & & & \oplus & & \\
 & & & & & & \mathcal{M}^3 & \xrightarrow{E} & \cdots
\end{array}
$$

Completing the picture

Thm (Brito, DM)

For each $d \geq 0$, the $U_{q^2}(\mathfrak{sl}_2)$-equivariant projection $\Pi_d : \mathcal{P}(\mathcal{S})^d \to \mathcal{M}^d$ is given by

$$\Pi_d = \sum_{j=0}^{d} \frac{[2d-j]_q!}{[j]_q![2d]_q!} E^j F^j.$$

Thm (Brito, DM)

The elements $\gamma_2^{-2}\partial_{1,q^2}, \partial_{2,q^2}$ are generalized symmetries of F such that

$$\gamma_2^{-2}\partial_{1,q^2}, \partial_{2,q^2} : \mathcal{M}^d \to \mathcal{M}^{d-1}.$$

and $\mathrm{span}\{\gamma_2^{-2}\partial_{1,q^2}, \partial_{2,q^2}\} \cong L_{q^2}(-1)$.

Completing the picture, pt. 2

- For any invertible $X \in \mathcal{W}_{2,q} \otimes \mathcal{C}_{s,q}$ let $\{X\}_q := \frac{X - X^{-1}}{q - q^{-1}}$
- Using $\mathrm{ad}_F(X) = FX - (K^{-1}XK)F$in $U_q(\mathfrak{sl}_2)$, let

$$Z_1 = [2]_q \mu_1 \{K\}_q \{q^{-1}K\}_q + [2]_q E\,\mathrm{ad}_F(\mu_1)\{q^{-1}K\}_q + E^2\,\mathrm{ad}_{F^2}(\mu_1)$$

$$Z_2 = [2]_q (\gamma_1^2 \mu_2) \{K\}_q \{q^{-1}K\}_q + [2]_q E\,\mathrm{ad}_F(\gamma_1^2 \mu_2)\{q^{-1}K\}_q + E^2\,\mathrm{ad}_{F^2}(\gamma_1^2 \mu_2)$$

Thm (Brito, DM)

The elements Z_1, Z_2 are generalized symmetries of F such that

$$Z_1, Z_2 : \mathcal{M}^d \to \mathcal{M}^{d+1}.$$

and $\mathrm{span}\{Z_1, Z_2\} \cong L_{q^2}(-1)$.

Higher rank difficulties

- Realizations in the Weyl part: [Zhang-Hu, '17]

$\mathfrak{sp}_4$ in $\mathcal{W}_4$	$U_q(\mathfrak{sp}_4) \twoheadrightarrow \mathcal{W}_{4,q}$
$E_{1,w} = \mu_1\partial_2 - \mu_3\partial_4$	$E_{1,w} = \mu_1\partial_2\gamma_3\gamma_4^{-1} - \mu_3\partial_4$
$F_{1,w} = \mu_2\partial_1 - \mu_4\partial_1$	$F_{1,w} = \mu_2\partial_1 - \mu_4\partial_3\gamma_2\gamma_1^{-1}$
$H_{1,w} = \mu_1\partial_1 - \mu_2\partial_4$	$K_{1,w} = \gamma_1\gamma_2^{-1}\gamma_3\gamma_4^{-1}$
$E_{2,w} = \mu_2\partial_3$	$E_{2,w} = \mu_2\partial_{3,q^2} + \frac{(q-q^{-1})}{[2]_q}\mu_1\mu_4\partial_3^2\gamma_2\gamma_3$
$F_{2,w} = \mu_3\partial_2$	$F_{2,w} = \mu_3\partial_{2,q^2} + \frac{(q-q^{-1})}{[2]_q}\mu_1\mu_4\partial_2^2\gamma_2\gamma_3$
$H_{2,w} = \mu_2\partial_2 - \mu_3\partial_3$	$K_{2,w} = \gamma_2^2\gamma_3^{-2}$

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Thank you!

EMAIL: ADMISSIONS@FORWARD-COLLEGE.EU TEL: +33.7.69.37.51.04
CITÉ INTERNATIONALE UNIVERSITAIRE, LA MAISON INTERNATIONALE,
17 BOULEVARD JOURDAN 75014 PARIS FRANCE
RUA DAS FLORES 71, 1200-213 LISBOA, PORTUGAL
KÖPENICKER STR. 43, 10179 BERLIN, GERMANY

AWARDING BODY

UNIVERSITY OF LONDON

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