A Remark on the Langlands Correspondence for Tori

(joint with E. Opdam)

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Outline

Fields and Tori: Basic Setup

- Throughout, F is a field with two "flavours":
  - **Global:** $\mathbb{Q}, \mathbb{F}_p(t)$ and finite extensions
  - **Local:** $\mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p[\![t]\!], \ldots$
  - For F global, consider $\mathbb{A}_F \subset \prod_v F_v$ and let

$$C_F := \begin{cases} F^*, & F \text{ local}, \\ \mathbb{A}_F^*/F^*, & F \text{ global}. \end{cases}$$

- T algebraic torus defined over F, splitting over E/F Galois, finite.
  - $X = \mathrm{Hom}(T, \mathbb{G}_m)$
  - $\Gamma = \mathrm{Gal}(E/F)$

Examples of Tori

- **Split torus:**

$$\mathbb{G}_m/F : (x, y) \mid xy = 1, \qquad F[\mathbb{G}_m] = F[x, x^{-1}]$$
$$X(\mathbb{G}_m) : x \mapsto x^n, \qquad X(\mathbb{G}_m) \cong \mathbb{Z}$$

- **Non-split torus:**

$$\mathbb{S}/\mathbb{R} : (x, y) \mid x^2 + y^2 = 1$$
$$\mathbb{S}/\mathbb{C} : (z, w) \mid zw = 1, \qquad z = x + (\sqrt{-1})y,\ w = x - (\sqrt{-1})y$$
$$X(\mathbb{S}) : z \mapsto z^n, \qquad \gamma\colon (z \mapsto z^n) \mapsto (z \mapsto z^{-n})$$

where $\Gamma = \langle \gamma \rangle$.

Duality for Tori

- **Borel's theorem:** There is an equivalence of categories

$$\left\{ \begin{array}{c} \text{Algebraic tori } T/F \\ \text{splitting over } E/F \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lattices } X \text{ with} \\ \Gamma\text{-action} \end{array} \right\}$$

$$T \quad \longmapsto \quad X = \mathrm{Hom}(T, \mathbb{G}_m)$$

- **Dual torus:** $T \longmapsto X \rightsquigarrow \widehat{X} = \mathrm{Hom}(X, \mathbb{Z})$ and

$$\widehat{T} = \mathrm{Hom}(\widehat{X}, \mathbb{C}^*).$$

Relative Weil group

- **Relative Weil group:** an extension[1]

$$1 \longrightarrow C_E \longrightarrow W_{E/F} \longrightarrow \Gamma \longrightarrow 1.$$

- **Example:** If $F = \mathbb{R}$ then $\Gamma = \{1, \gamma\}$ and

$$0 \to \mathbb{C}^* \to W_{\mathbb{C}/\mathbb{R}} \to \Gamma \to 0,$$

with

$$W_{\mathbb{C}/\mathbb{R}} = \mathbb{C}^* \sqcup j\mathbb{C}^*,$$

and

$$jzj^{-1} = \bar{z}, \qquad j^2 = -1 \in \mathbb{C}^*.$$

[1] determined by the fundamental class in $H^2(\Gamma, C_E)$

Langlands' Correspondence for Tori

- Recall the objects T, Γ, $W_{E/F}$ and $\widehat{T}$ as above.
- Define

$$\mathsf{X}(T,F) := \begin{cases} \mathrm{Hom}_c(T(F), \mathbb{C}^*), & F \text{ local}, \\ \mathrm{Hom}_c(T(\mathbb{A}_F)/T(F), \mathbb{C}^*), & F \text{ global}. \end{cases}$$

Langlands' Theorem

There is a canonical homomorphism

$$\Lambda\colon H^1_c(W_{E/F}, \widehat{T}) \longrightarrow \mathsf{X}(T,F)$$

(a) Λ is an isomorphism if F local.

(b) Λ is finite to one if F global.

Main Result of This Talk

Theorem (De Martino–Opdam)

Equip $Z_c^1(W_{E/F}, \widehat{T}), B_c^1(W_{E/F}, \widehat{T})$ and $\mathrm{X}(T, F)$ with the compact-open topology. Equip $H_c^1(W_{E/F}, \widehat{T})$ with the quotient topology. Then:

(1) The spaces $H_c^1(W_{E/F}, \widehat{T})$ and $\mathrm{X}(T, F)$ are abelian Lie groups.

(2) Langlands' homomorphism

$$\Lambda\colon H_c^1(W_{E/F}, \widehat{T}) \longrightarrow \mathrm{X}(T, F)$$

is an open homomorphism of Lie groups.

Outline

Setup: groups, modules, and the topology on cochains

- G: locally compact, second countable group (with a fixed Haar measure).
- A: Polish abelian group with continuous G-action.
  - Polish $=$ separable completely metrizable.
- Measurable cochains:

$$\underline{C}^n(G,A) := \{\text{Borel measurable maps } G^n \to A\}/\text{a.e.}$$

- Topology: convergence in measure.
- Differentials $\underline{\delta}^n : \underline{C}^n(G,A) \to \underline{C}^{n+1}(G,A)$ with usual alternating formula.
- Define

$$\underline{Z}^n(G,A) := \ker(\underline{\delta}^n),\ \underline{B}^n(G,A) := \mathrm{im}(\underline{\delta}^{n-1}),\ \underline{H}^n(G,A) := \underline{Z}^n(G,A)/\underline{B}^n(G,A).$$

- All groups are given the subspace/quotient topologies induced from $\underline{C}^n(G,A)$.

Moore Cohomology Theory Summary

Theorem (Moore, Austin-Moore)

(a) $\underline{\delta}^n$ continuous for all n.

(b) If $\underline{B}^n(G,A) \subset \underline{Z}^n(G,A)$ closed, then $\underline{H}^n(G,A)$ is Polish.

(c) $\underline{H}^1(G,A) \cong H^1_c(G,A)$ if Z^1_c, B^1_c compact-open and H^1_c quotient topologies.

(d) Given $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of Polish G-modules all morphisms in the long exact sequence induced by $\underline{H}^*(G,-)$ are continuous.

(e) A pair $(G \to G', A' \to A)$ induces continuous $\underline{H}^p(G',A') \to \underline{H}^p(G,A)$ for all p.

(f) Suppose that the quotient G/G° is compact.
   - A discrete $\implies$ $\underline{H}^p(G,A)$ is countable and discrete for all $p > 0$.
   - $A = \mathbb{R}^n \implies \underline{H}^p(G,A)$ a finite-dimensional real vector space for all $p > 0$.

(g) If G is compact and $A = \mathbb{R}^n$, then $\underline{H}^p(G,A) = 0$, for all $p > 0$.

Outline

Roadmap: what we use

- Long-exact sequence starting from $0 \to X \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 0$.
- Recall (Inflation-Restriction): a SES $0 \to N \to G \to Q \to 0$ yields
$$0 \to \underline{H}^1(Q, A^N) \to \underline{H}^1(G, A) \to \underline{H}^1(N, A)^Q \to \underline{H}^2(Q, A^N) \to \underline{H}^2(G, A)$$
- Inflation-Restriction with
$$0 \to W^1_{E/F} \to W_{E/F} \to V_F \to 0,$$
where $W^1_{E/F}$ compact and $V_F = \mathbb{R}$ or $V_F = \mathbb{Z}$.
- Examples:
$$F = \mathbb{C} \implies \|\cdot\| : \mathbb{C}^* \to e^{\mathbb{R}} \text{ with compact kernel.}$$
$$F = \mathbb{Q}_p \implies \|\cdot\| : \mathbb{Q}_p^* \to p^{\mathbb{Z}} \text{ with compact kernel.}$$

(1) First computation: $\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))$

- Moore vanishing:
$$\underline{H}^1(W^1_{E/F}, \mathrm{Lie}(\widehat{T})) = 0.$$
- Inflation–restriction from $0 \to W^1_{E/F} \to W_{E/F} \to V_F \to 0$ gives
$$0 \to \underline{H}^1(V_F, \mathrm{Lie}(\widehat{T})^{W_{E/F}}) \to \underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \to \underline{H}^1(W^1_{E/F}, \mathrm{Lie}(\widehat{T}))^{V_F} = 0$$
- $\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \cong \underline{H}^1\left(V_F, \mathrm{Lie}(\widehat{T})^{W^1_{E/F}}\right).$
- $V_F = \mathbb{R} \implies \underline{H}^1\left(V_F, \mathrm{Lie}(\widehat{T})^{W^1_{E/F}}\right) \cong \mathrm{Hom}_c(V_F, \mathrm{Lie}(\widehat{T})^\Gamma) \cong \mathrm{Lie}(\widehat{T})^\Gamma.$
- $V_F = \mathbb{Z} \implies \underline{H}^1\left(V_F, \mathrm{Lie}(\widehat{T})^{W^1_{E/F}}\right) \cong \mathrm{Lie}(\widehat{T})^\Gamma.$

(2) Discrete pieces: $\underline{H}^1(W_{E/F}, X)$ and $\underline{H}^2(W_{E/F}, X)$

- X is countable discrete (free abelian).
- If $V_F = \mathbb{R} \implies W_{E/F}/W^{\circ}_{E/F}$ is compact.

  $\underline{H}^1(W_{E/F}, X)$, $\underline{H}^2(W_{E/F}, X)$ are countable and discrete (quotient topology).

- Also true for $V_F = \mathbb{Z}$, but needs more work.

(3) Identity Component: The $V_F = \mathbb{R}$ case.

- Inflation–restriction gives an exact sequence (continuous arrows):

$$0 \to \underline{H}^1\left(\mathbb{R}, \widehat{T}^{W^1_{E/F}}\right) \to \underline{H}^1(W_{E/F}, \widehat{T}) \to \underline{H}^1(W^1_{E/F}, \widehat{T}).$$

- The right term is discrete:
  - From $0 \to X \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 0$ we have

$$\underline{H}^1(W^1_{E/F}, \mathrm{Lie}(\widehat{T})) \to \underline{H}^1(W^1_{E/F}, \widehat{T}) \to \underline{H}^2(W^1_{E/F}, X) \to \underline{H}^2(W^1_{E/F}, \mathrm{Lie}(\widehat{T}))$$

- Conclusion:

$$\underline{H}^1(W_{E/F}, \widehat{T})^\circ \cong H^1_c(W_{E/F}, \widehat{T})^\circ \cong \mathrm{Lie}(\widehat{T})^\Gamma.$$

(4) Identity Component: The $V_F = \mathbb{Z}$ case.

- Exponential short exact sequence:

$$0 \to X \to \mathrm{Lie}(\widehat{T}) \to \widehat{T} \to 0.$$

- Long exact sequence (continuous arrows):

$$\underline{H}^1(W_{E/F}, X) \to \underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \to \underline{H}^1(W_{E/F}, \widehat{T}) \to \underline{H}^2(W_{E/F}, X).$$

- From previous slides:

$$\underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T})) \approx \mathrm{Lie}(\widehat{T})^\Gamma, \qquad \underline{H}^1, \underline{H}^2 \text{ with } X \text{ are discrete.}$$

(4b) Identity Component: The $V_F = \mathbb{Z}$ case.

- Let
$$A := \mathrm{im}\left(\underline{H}^1(W_{E/F}, X) \to \underline{H}^1(W_{E/F}, \mathrm{Lie}(\widehat{T}))\right) \subset \mathrm{Lie}(\widehat{T})^\Gamma.$$
- A is countable and closed $\Rightarrow$ discrete in a finite-dimensional vector space and
$$0 \to A \to \mathrm{Lie}(\widehat{T})^\Gamma \to H^1_c(W_{E/F}, \widehat{T})^\circ \to 0$$
- A is in fact a lattice in the real form $\mathbb{R} \otimes X^\Gamma \subset \mathrm{Lie}(\widehat{T})^\Gamma$.

Moral of the story

- The character side $\mathrm{X}(T,F)$ is known to be an abelian Lie group:

$$0 \to L_F \to \mathrm{Lie}(\widehat{T})^\Gamma \to \mathrm{X}(T,F)^\circ \to 0$$

  with $L_F = 0$ if $V_F = \mathbb{R}$ and L_F a lattice if $V_F = \mathbb{Z}$.
- On the other hand, our constructions yield

$$0 \to A_F \to \mathrm{Lie}(\widehat{T})^\Gamma \to H^1_c(W_{E/F}, \widehat{T})^\circ \to 0$$

  with $A_F = 0$ if $V_F = \mathbb{R}$ and A_F a lattice if $V_F = \mathbb{Z}$.

Moral of the story

- From the commutative diagram implied from Langlands constructions

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
A_F \arrow[r] \arrow[d] & L_F \arrow[d] \\
\mathrm{Lie}(\widehat{T})^\Gamma \arrow[r] \arrow[d] & \mathrm{Lie}(\widehat{T})^\Gamma \arrow[d] \\
H^1_c(W_{E/F}, \widehat{T})^\circ \arrow[r, "\Lambda"] \arrow[d] & \mathrm{X}(T, F)^\circ \arrow[d] \\
0 & 0
\end{tikzcd}$$

Moral of the story

- We prove linearity on the middle row.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] \\
 & A_F \arrow[rr] \arrow[d] & & L_F \arrow[d] \\
 & H^1_c(W_{E/F}, \mathrm{Hom}(\widehat{X}, \mathbb{C})) \arrow[rr, "\mathrm{d}\Lambda"] \arrow[d] & & \mathrm{Hom}_c(\mathrm{Hom}_\Gamma(X, C_E), \mathbb{C}) \arrow[d] \\
 & H^1_c(W_{E/F}, \widehat{T})^\circ \arrow[rr, "\Lambda"] \arrow[d] & & \mathrm{X}(T, F)^\circ \arrow[d] \\
 & 0 & & 0
\end{tikzcd}
$$

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Thank you!

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