Dirac operators for the Dunkl Angular Momentum Algebra

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Outline

Dirac Operators in Representation Theory

- $\mathcal{A}$ associative algebra, $\mathcal{C}$ a Clifford algebra and V a vector space

$$\{v_1, \ldots, v_n\} \subset V \subset \mathcal{A} \cap \mathcal{C} \rightsquigarrow \mathcal{D} = \sum_i v_i \otimes v_i^* \in \mathcal{A} \otimes \mathcal{C}$$

- Typically, $\mathcal{D}^2$ is "nice":
  - $\mathcal{A} = \mathrm{Weyl}(\mathbb{R}^n)$, $\mathcal{C} = \mathrm{Cliff}(\mathbb{R}^n)$

$$\mathcal{D} = \sum_i \partial_i \otimes e_i \Rightarrow \mathcal{D}^2 = \Delta.$$

  - $\mathcal{A} = \mathcal{U}(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, $\mathcal{C} = \mathrm{Cliff}(\mathfrak{p}, \kappa)$

$$\mathcal{D} = \sum_i Z_i \otimes Z_i \Rightarrow \mathcal{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + \mathrm{cst}.$$

- Dirac operators $\mathcal{D}$ have a very successful history in Lie theory
  - Lie groups: Atyiah-Schmid, Parthasaraty.
  - $(\mathfrak{g}, K)$-modules: Vogan, Huang-Pandzic [HP].
  - Drinfeld algebras: Barbasch, Ciubotaru, Opdam, Trapa.

Goals

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- Dirac operators when $\mathcal{A}$ is the Dunkl-Angular Momentum Algebra.
  - Formula for the square.
  - Analogies with Vogan Conjectures:

$$\mathcal{U}(\mathfrak{g})\text{-Mod} \longleftrightarrow \tilde{K}\text{-Mod}$$

Outline

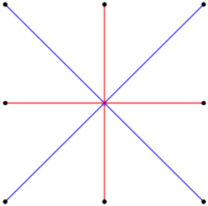
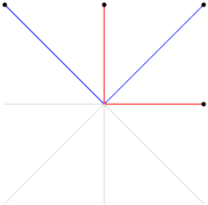
The Classical Harmonic Duality

- Harmonic $\mathfrak{sl}_2$: $\mathfrak{a} = \mathrm{Lie}(\Delta, |x|^2)$, in $\mathrm{Weyl}(\mathbb{R}^n)$

$$\begin{tikzcd}
\mathcal{P}^0 & \mathcal{P}^1 & \mathcal{P}^2 & \mathcal{P}^3 & \cdots \\
\mathcal{H}^0 \arrow[rr, "|x|^2"] & & |x|^2\mathcal{H}^0 \arrow[rr, "|x|^2"] & & \cdots \\
& \mathcal{H}^1 \arrow[rr, "|x|^2"] & & |x|^2\mathcal{H}^1 \arrow[r] & \cdots \\
& & \mathcal{H}^2 \arrow[rr, "|x|^2"] & & \cdots \\
& & & \mathcal{H}^3 \arrow[r] & \cdots
\end{tikzcd}$$

- AMA: $A := \mathrm{Cent}_{\mathrm{Weyl}(\mathbb{R}^n)}(\mathfrak{a}) \supset \mathrm{span}\{L_{ij} := \mu_i\partial_j - \mu_j\partial_i\}_{i<j} \twoheadleftarrow \mathcal{U}(\mathfrak{so}_n)$

Dunkl operators

- Let $\mathbb{R}^n \supset R$:  $\supset R^+$: 
- Let $G = G(R) \subset O(n)$, a finite reflection group.
- Fix $c : R^+ \to \mathbb{C}$, $\quad \alpha \mapsto c_\alpha \quad$ (with $c_{g\alpha} = c_\alpha$ for $g \in G$).
- Replace

$$\partial_j \to D_{c,j} = \partial_j + \sum_{\alpha \in R^+} c_\alpha (e_j, \alpha^\vee) \frac{\mathrm{Id} - s_\alpha}{\alpha}.$$

Rational Cherednik algebra and the Dunkl-AMA

- $\mathsf{H}_c = \mathsf{H}(G, c) = \mathsf{Alg}(\{D_{c,j}\}, \{\mu_j\}, G) \subset \mathsf{End}(\mathcal{P})$

$$\mathsf{H}_c = S(D_{c,1}, \ldots, D_{c,n}) \otimes S(\mu_1, \ldots, \mu_1) \otimes \mathbb{C}[G]$$
$$[D_{c,j}, \mu_k] = [\partial_j, \mu_k] + \textstyle\sum_{\alpha \in R^+} c_\alpha (e_j, \alpha^\vee)(\alpha, e_k) s_\alpha$$
$$\sum_i [D_{c,i}, \mu_i] = n + 2Z_c$$

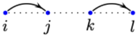
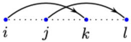
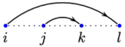
with $Z_c = \sum_{\alpha \in R^+} c_\alpha s_\alpha \in Z(\mathbb{C}[G])$.

- Get Δ_c and $\mathfrak{sl}_2 \cong \mathfrak{a}_c = \mathsf{Lie}(\Delta_c, |x|^2) \subset \mathsf{H}_c$ and the Dunkl-AMA:

$$\mathsf{A}_c := \mathsf{Cent}_{\mathsf{H}_c}(\mathfrak{a}_c) \supset \mathsf{span}\{L_{c,ij} := \mu_i D_{c,j} - \mu_j D_{c,i}\}_{i<j}$$

Properties

- No more a copy of $\mathfrak{so}_n$ in A_c.
- [FH]: $\mathsf{A}_c = \mathrm{Alg}(\{L_{c,jk}\}, G)/$crossing relations.

$L_{c,ij} =$ ,  $-$  $+$  $\equiv 0$

- Representation theory of H_c (and A_c) are sensitive to c.

Outline

The AMA Dirac Element and its Square

- As a vector space $\mathfrak{so}_n = \wedge^2(\mathbb{R}^n)$. Consider $\mathcal{C} = \mathrm{Cliff}(\mathbb{R}^n)$

$$\begin{array}{cccc} \mathsf{A}_c & \supset & \mathrm{span}\{L_{c,ij}\}_{i<j} & \cong & \wedge^2(\mathbb{R}^n) \\ \mathcal{C} & \supset & \mathrm{span}\{e_ie_j\}_{i<j} & \cong & \wedge^2(\mathbb{R}^n). \end{array}$$

- Set

$$\begin{aligned} \mathcal{D} &= \sum_{i<j} L_{c,ij} \otimes e_ie_j \\ \mathfrak{D}_0 &= \mathcal{D} - \phi && (\phi = \tfrac{1}{2}(2Z_c - n + 2)) \\ \Rightarrow \mathfrak{D}_0^2 &= \Omega_{\mathfrak{sl}_2} + \mathrm{cst} && (\text{see [CDM]}) \end{aligned}$$

- Meaningful as, essentially, $Z(\mathsf{A}_c) = \mathbb{C}[\Omega_{\mathfrak{sl}_2}]$. True if $(-\mathrm{Id})_{\mathbb{R}^n} \notin G$.

$$\text{if } G \ni w_0 = (-\mathrm{Id})_{\mathbb{R}^n} \Rightarrow Z(\mathsf{A}_c) = \mathbb{C}[\Omega_{\mathfrak{sl}_2}] \cup \mathbb{C}[\Omega_{\mathfrak{sl}_2}]w_0$$

Analogue of Vogan's Conjecture

- Have $1 \to \mathbb{Z}_2 \to \tilde{G} \xrightarrow{p} G \to 1$ and $\rho = p \otimes \mathrm{Id} : \tilde{G} \to \mathsf{A}_c \otimes \mathcal{C}$.
- Call $C \in Z(\mathbb{C}[\tilde{G}])$ with $C^* = C$ an **admissible element**. For such C

$$\mathfrak{D}_C = \mathfrak{D}_0 + \rho(C).$$

Theorem (Calvert, DM)

For each admissible $C \in Z(\mathbb{C}[\tilde{G}])$ there is an algebra morphism

$$\zeta_C : Z(\mathsf{A}_c) \to Z(\mathbb{C}[\tilde{G}])$$

such that for all $z \in Z(\mathsf{A}_c)$ there is $a \in \mathsf{A}_c \otimes \mathcal{C}$ such that

$$z \otimes 1 = \rho(\zeta_C(z)) + \mathfrak{D}_C a + a\mathfrak{D}_C.$$

Further results pt. 1

- Fix C admissible and (π, X) an A_c-module (and $(\sigma, \mathbb{S})$ a spin $\mathcal{C}$-mod)
  $$\rightsquigarrow D_C = (\pi \otimes \sigma)(\mathfrak{D}_C) \in \mathrm{End}(X \otimes \mathbb{S})$$

Definition (Dirac Cohomology)

The Dirac cohomology is the $\tilde{G}$-module $H(X, C) = \frac{\ker(D_C)}{\ker(D_C) \cap \mathrm{Im}(D_C)}$.

Definition

If $\tilde{\tau} \in \mathrm{Irr}(\tilde{G})$ let $\chi_{\tilde{\tau}} : Z(\mathsf{A}_c) \to \mathbb{C}, \quad \chi_{\tilde{\tau}}(z) = \frac{1}{\dim \tilde{\tau}} \mathrm{Tr}(\tilde{\tau}(\zeta_C(z)))$.

Theorem (Calvert, DM)

Let X be an A_c-module with $zx = \chi(z)x$ for all $z \in \mathsf{A}_c$ and $x \in X$. Then

$$\mathrm{Hom}_{\tilde{G}}(\tilde{\tau}, H(X, C)) \neq 0 \Rightarrow \chi = \chi_{\tilde{\tau}}.$$

Further results pt. 2

Theorem (Calvert, DM)

When $G = S_n$, any admissible is expressed as $C = C_1 + iC_2$ with

$$C_j \leftrightarrow \{\sigma(M_1^2, \cdots, M_n^2) \mid \sigma \text{ symmetric polynomial}\}$$

where $M_1, \ldots M_n$ are the Jucys-Murphy elements.

Theorem (Calvert, DM)

When $G = S_n$, if X is a unitary module of A_c such that $\mathrm{spec}(\Omega_{\mathfrak{sl}_2}) \geq -1$, then there is C such that $H(X, C) \neq 0$.

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Thanks for the attention!

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[HP] J.-S. Huang, P. Pandzic
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